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SPHERICALLY SYMMETRIC, POLYTROPIC FLOW

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ABSTRACT

We determine, in closed form, the flow profile corresponding to stationary, spherically symmetric, polytropic flow around a gravitating and radiating point mass. The profile is a branch of a quartic equation, relating distance to the central object to, for example, the local sound speed. Two branches of this equation are complex. The other two describe three different types of flow: subsonic and transonic accretion flow, transonic wind, and supersonic, super-Eddington flow. At any position in this flow, a supersonic regime may be linked to a subsonic one, through an adiabatic shock. Given the profiles, we determine the position of this shock, and vice versa. We also derive a stability criterion for the continuous and shocked flows.

Subject headings: accretion, accretion disks — hydrodynamics — shock waves — stars: mass-loss

1. INTRODUCTION

The problem of spherical polytropic flow has entered the literature through the study of both stellar wind (Parker 1963, 1966) and accretion (Bondi 1952). In both cases it represents only a simple model for a complex phenomenon for which spherical symmetry as well as the assumption of a polytropic equation of state is mostly a strong approximation. Nevertheless spherical polytropic flow is an interesting topic for study, both in its own right and because it helps to understand some of the basic features of stellar accretion and outflow. An important advantage is the fact that the hydrodynamical equations for this model can be handled without recourse to purely numerical solutions or simulations. In the present paper we reconsider the problem of spherical polytropic flow from a more general point of view and carry its analytic treatment a little further than has been done hitherto. In order to keep in mind its application to the description of accretion and outflow, we give a brief review of the literature on both topics.

When a collapsed object of mass M (a star, black hole, galaxy, etc.—we shall call this object a star in what follows) moves with velocity v_∞ through polytropic gas with density ρ_∞, it will accrete matter at a rate $\dot{M} = (\pi R_A^2)\rho_\infty v_\infty$ (in all equations in this paper we put the gravitational constant $G = 1$). If the accretion radius R_A is much larger than the radius of the sphere, $R_A \gg R_S$, this leads to the following expression, first derived by Eddington (1926):

$$\dot{M}_{\rm E} = 2\pi M R_S \rho_\infty / v_\infty \; . \tag{1.1}$$

Gas particles with impact parameter larger than R_A will be deflected round the star. If the accretion is axisymmetric, the particle can collide with a particle passing at the other side of the star at the symmetry line (the accretion line). This collision will cancel their tangential velocity with respect to the star. If their radial velocity is smaller than the local escape speed from the star, they will be accreted by the star (from behind). This increases the accretion rate above the value (e.g. [1.1] to (Hoyle & Lyttleton 1939; Bondi & Hoyle 1944):

$$\dot{M}_{\rm HL} = 2.5\pi M^2 \rho_\infty / v_\infty^3 \; . \tag{1.2}$$

This expression diverges for $v_\infty \to 0$, due to the fact that we have neglected the buildup of a pressure gradient in deriving equation (1.2). Bondi (1952) derived the accretion rate for the case $v_\infty = 0$, but including the pressure gradient. He obtained the following expression:

$$\dot{M}_{\rm B} = 2\pi\lambda M^2 \rho_\infty / c_\infty^3 \; , \tag{1.3}$$

where c_∞ is the sound speed at infinity. The dimensionless accretion rate λ is not determined by the model. Bondi showed that there are two possible types of stationary flow. If $\lambda < \lambda_{\rm max}$, where $\lambda_{\rm max}$ depends only on the value of the polytropic exponent γ, the flow is subsonic everywhere ($v < c$). For $\lambda = \lambda_{\rm max}$, a transonic solution exists, with Mach number going to infinity close to the star. For $\lambda > \lambda_{\rm max}$, no stationary solution exists. The transonic solution with $\lambda = \lambda_{\rm max}$ is usually referred to as the Bondi solution.

No analytic treatment of the problem with $v_\infty > 0$ and including gas pressure exists. Bondi (1952) combined expressions (1.2) and (1.3) and suggested

$$\dot{M}_{\rm BH} = 2\pi M^2 \frac{\rho_\infty}{(v_\infty^2 + c_\infty^2)^{3/2}} \; , \tag{1.4}$$

as a generalization. The form in equation (1.4) has the correct limiting behavior for $v_\infty \to 0$ or $c_\infty \to 0$. It gives the order of magnitude of the accretion rate in this case, as was verified numerically by Hunt (1971).

The accretion rates (1.2)–(1.4) have been applied to a wide range of systems. The late stages of stellar collapse, after the formation

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of a dense core, are represented well by the accretion rate (1.3) and its associated flow profile (see, e.g., Boss & Black 1982; Zinnecker & Tscharnuter 1983). Bhattacharjee (1987) used this to study pre-main-sequence evolution. Bhattacharjee & Williams (1980) used equation (1.4) to derive a theoretical expression for the initial mass function of stars, forming in a dense cloud, after the passage of a shock wave.

The rate (1.4) has been applied to study the accretion onto a white dwarf star, moving through the interstellar medium (Yoshi 1981; Greenstein 1982, 1986; Iben & Tutukov 1984; Wegner & Yackovich 1984; Wood & Faulkner 1986). Equation (1.2) has been applied to calculate the accretion rate of a binary star, accreting from a mass-losing companion (Livio & Soker 1983, 1988; Nussbaumer & Vogel 1987).

To get beyond the simplifying assumptions of the analytic treatment, many detailed numerical investigations were done (e.g., Hunt 1971; Fryxell, Taam, & McMillan 1987; Livio et al. 1986; Ho et al. 1989, and references therein). Although the flow profiles in the case $v_\infty > 0$, $c_\infty > 0$ are generally very complex, expression (1.4) is usually accurate to within a factor 2–3. Most recent calculations have concentrated on the accretion of angular momentum.

We will now turn our attention to the study of outflow. The equations for accretion and outflow differ only in the imposed boundary conditions, yet, both fields seem to have developed quite separately.

In a classical set of papers, Parker (1958, 1963, 1965) established the basis for the theory of the solar wind as a hydrodynamic expansion of the solar corona. In Parker's description, a polytropic gas expands along "flow tubes" for which the cross section $A(l)$ depends on the distance along the flow tube l like $A(l) \propto (l/a)^s$, where s and a are constants. Spherically symmetric flow is represented by $s = 2$. The flow tubes are the mathematical description of coronal holes. The solution, for a given value γ of the polytropic index, is uniquely determined by values at the surface of the star, unlike the case for accretion. This is because for wind, the only physically acceptable solution is the transonic one which means that the outflow rate is determined by the model. Parker pointed out that there is no physically acceptable solution (with flow speed decreasing to zero, close to the star) for $\gamma > 3/2$. He furthermore studied the shock which occurs as the solar wind encounters the interstellar medium.

The addition of extra physics, like a time- and/or distance-dependent cross section $A(l, t)$ or, equivalently, the addition or subtraction of momentum to the flow (Bailyn, Rosner, & Tsinganos 1985) may drastically alter the characteristics of the flow through the introduction of more critical points (Kopp & Holzer 1976; Holzer 1977; Habbal & Tsinganos 1983). This may lead to degeneracy, that is, different flow profiles still satisfy the same boundary conditions. The effects of the inclusion of magnetic fields (e.g., Low 1985; Sakurai 1985; Marsch 1986) and cosmic rays (e.g., Ko & Webb 1988) were investigated.

The model was also applied to the study of astrophysical jets (see, e.g., Trussoni et al. 1988).

In this paper, we will derive the flow profile in closed form for the spherical case, as the solution of a quartic equation. This allows better insight into the properties of the flow, especially in the vicinity of the sonic point ($v = c$). We recover the three known types of spherical polytropic flow:

1. Completely subsonic flow, which may occur in the case of accretion, depending on the conditions prevailing at the star (Petterson, Silk, & Ostriker 1980, hereafter PSO), but not in the case of stellar wind, because it would require too large an inward pressure from the surrounding interstellar medium (Parker 1963);
2. Transonic flow which is relevant both for winds and for accretion;
3. "Super-Eddington" flow, which occurs if the net force is outward. This may occur if the luminosity of the central star exceeds the Eddington limit and the continuum radiation pressure dominates the gravitational force. For line-driven winds (Castor, Abbott, & Klein 1975), the radiation force goes like $\sigma(r, v, dv/dr)/r^2$, where the $1/r^2$ term describes the effect of geometrical dilution of the radiation, and $\sigma(r, v, dv/dr)$ describes (partial) absorption. According to Castor et al. (1975), in the supersonic part of the flow profile, due to the redshift of the outflowing gas, σ is approximately constant; then the net force may still be outward and behave as $1/r^2$, even when the luminosity is below the Eddington limit. In what follows, we will still term this kind of flow as "super-Eddington" flow.

The closed form solution thus provides in particular simplification and unification of several previous theoretical treatments of spherical polytropic flow. We use the solution to obtain asymptotic expressions for the sub- and supersonic profiles (at $r = 0$, $r = \infty$, and at the sonic point), which were already obtained previously, and also for the super-Eddington case, for which, to the best of our knowledge, they were not obtained before.

We will discuss the stability of these profiles for linear, spherically symmetric perturbations. The stability of Parker's transonic solution was investigated in consecutive papers: Parker (1966) argued that the solution is stable, but Carovillano & King (1966) claimed that the solution was unstable with respect to a certain class of (linear) perturbations. Balazs (1972) found the accretion flow to be unstable in the isothermal case. This result was shown to be misguided by Stellingwerf & Buff (1978), who investigated the problem numerically. They showed that the transonic solution is stable, also in the wind case, but the subsonic solutions are not. Using energy considerations, Garlick (1979) showed that the subsonic solution is generally stable, depending on the inner boundary conditions on the perturbation. He also studied nonspherical perturbations. PSO came to the same conclusions, using the WKB approximation, to study the behavior of high-frequency running wave perturbations. They also showed that subsonic solutions are stable to standing waves.

We show that the possible perturbations are solutions of a Sturm-Liouville eigenvalue problem in any interval not containing the sonic point or a shock. We also make more precise the constraints on the inner boundary condition to have stability. The transonic solution is stable.

In § 2, we give the equations for spherically symmetric, polytropic accretion and outflow. We reduce this set of coupled differential equations to one quartic equation. We use this equation to study the nature of the solutions and to derive asymptotic expressions. In § 3, we discuss the stability of these profiles. The results are summarized in § 4.

2. FLOW PROFILE

2.1. *Continuous Profile*

2.1.1. *Closed Form Solution*

The time-dependent equations, describing spherically symmetric, polytropic flow are:

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} = -\rho \frac{\partial v_r}{\partial r} - 2 \frac{\rho v_r}{r} , \tag{2.1a}$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{f}{r^2} , \tag{2.1b}$$

$$p = \kappa \rho^{\gamma} , \tag{2.1c}$$

where ρ is the gas density, p the pressure, and v_r the radial velocity ($v_r > 0$ for outward motion). In equation (2.1c) κ is a constant, and γ is the polytropic exponent, for example, $\gamma = 1$ for isothermal flow, $\gamma = 4/3$ for radiation-dominated flow, and $\gamma = 5/3$ for an ideal gas. Equations (2.1a)–(2.1c) are the continuity equation, Euler's equation, and the polytropic equation, respectively. In equation (2.1b) f describes the combined effects of gravity and radiation pressure:

$$f = m + \int_0^r 4\pi r^2 \rho(r) dr - \frac{L}{L_E} m . \tag{2.2}$$

The first term in equation (2.2) describes the gravitational pull of the central star with mass m. This m is constant in time. The second term is the self-gravity contribution to the gravitational force. We will neglect this term. Chia (1978, 1979) included self-gravity and showed it leads to enhanced accretion. For a study of corrections due to rotation on the spherically symmetric case, we refer to Cassen & Pettibone (1976) and Cassen (1978). The last term in equation (2.2) comes from the radiation pressure. Here L is the central luminosity and L_E the Eddington limit:

$$\frac{L_E}{L_\odot} = 3.4 \times 10^4 \frac{m}{M_\odot} . \tag{2.3}$$

For $L > L_E$, the net force is outward. Some cases where the radiation pressure is not due to continuum absorption, but to line absorption, may also be approximated by putting $L > L_E$ in equation (2.2). With the neglect of self-gravity, f is a constant.

We now specialize to stationary solutions to equation (2.1). In that case, the equations (2.1a)–(2.1c) do not depend on the sign of v_r, so as far as stationary solutions are concerned, we will only consider $v = |v_r|$ and distinguish between accretion and outflow only in the interpretation. We can integrate equation (2.1a) directly to obtain:

$$4\pi r^2 \rho v = A \tag{2.4}$$

where the constant A is the accretion or mass-loss rate. Equation (2.1b) can also be integrated directly, together with equation (2.1c), to give Bernoulli's equation:

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{f}{r} = \text{constant} , \quad \text{for } \gamma > 1 \tag{2.5a}$$

$$\frac{1}{2} v^2 + c^2 \ln \rho - \frac{f}{r} = \text{constant} , \quad \text{for } \gamma = 1 , \tag{2.5b}$$

where c is the isothermal sound speed. We will treat the isothermal case separately in the Appendix. We write the constant in equation (2.5) as $c_*^2/(\gamma - 1)$. We will use c_* as our unit of velocity and $\rho_* \equiv \rho_0 (c_*/c_0)^{2/(\gamma-1)}$ as our unit of density, where ρ_0 and c_0 are density and sound speed at any chosen point in the flow (the sound speed c is not constant in eq. [2.5a]). For accretion flow, c_* and ρ_* are the values of sound speed and density at inifinity. We now define dimensionless variables c, r, v, and ρ by

$$c \to c c_* , \qquad r \to r \frac{f}{c_*^2} , \qquad v \to v c_* , \qquad \rho \to \rho \rho_* \tag{2.6}$$

to get

$$\frac{1}{2} v^2 + \frac{c^2 - 1}{\gamma - 1} - \frac{1}{r} = 0 \tag{2.7}$$

for equation (2.5a) and

$$r^2 \rho v = \lambda \equiv \frac{A c_*^3}{4\pi f^2 \rho_*} \tag{2.8}$$

for equation (2.4), where λ is the dimensionless accretion (or outflow) rate. Note that the dimensionless distance r will be negative when $L > L_E$. We now use equation (2.8) to eliminate v in equation (2.7) and use the relation $c^2 = \rho^{\gamma-1}$ to obtain

$$r^4 - \frac{\gamma-1}{z-1} r^3 = -\frac{1}{2} \lambda^2 \frac{\gamma-1}{(z-1)z^{2/(\gamma-1)}} , \tag{2.9}$$

where we have defined $z \equiv c^2$. This is a quartic equation for r as a function of z, describing the complete flow profile as a function of λ. Given $z = z(r)$, we find the corresponding profiles for c, v, ρ, and Mach number $M \equiv v/c$ from

$$c = \sqrt{z} , \tag{2.10a}$$

$$v = \frac{\lambda}{r^2 z^{1/(\gamma-1)}} , \tag{2.10b}$$

$$\rho = z^{1/(\gamma-1)} , \tag{2.10c}$$

$$M = \frac{\lambda}{r^2 z^{(\gamma+1)/2(\gamma-1)}} . \tag{2.10d}$$

For later use we also quote the following relation, which is obtained from equations (2.9) and (2.10d) by eliminating z,

$$\lambda^2 = M^2 r^4 \left[\frac{1 + (\gamma-1)/r}{1 + M^2(\gamma-1)/2}\right]^{(\gamma+1)/(\gamma-1)} , \tag{2.11a}$$

and its differential,

$$\frac{d\lambda^2}{\lambda^2} = \frac{2(1-M^2)}{2 + (\gamma-1)M^2} \frac{dM^2}{M^2} - 4 \frac{(5-3\gamma)/4 - r}{(\gamma - 1 + r)} \frac{dr}{r} . \tag{2.11b}$$

To obtain the four branches of equation (2.9), we transform r according to

$$r = a(s+1) \tag{2.12a}$$

with

$$a = \frac{1}{4} \frac{\gamma-1}{z-1} \tag{2.12b}$$

to get

$$s^4 - 6s^2 - 8s - 3 + \frac{2\lambda^2}{a^3 z^{2/(\gamma-1)}} = 0 . \tag{2.13}$$

Notice that, for $z > 1$, the resolvent cubic equation of equation (2.13) has only one real root y_3 when

$$\lambda^2 \le \lambda^2_{\max} = \left(\frac{1}{2}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{5-3\gamma}{4}\right)^{-(5-3\gamma)/(\gamma-1)} , \tag{2.14}$$

while for $z < 1$ it has only one real root whatever the value of λ. The real root can be written as

$$y_3 = -2 + \sqrt[3]{|R|} \left(\sqrt[3]{-\frac{R}{|R|} + \sqrt{1 + \frac{R}{216}}} + \sqrt[3]{-\frac{R}{|R|} - \sqrt{1 + \frac{R}{216}}}\right) \tag{2.15a}$$

where

$$R = -\frac{16\lambda^2}{a^3 z^{2/(\gamma-1)}} . \tag{2.15b}$$

The four branches of equation (2.9) are given by

$$r_1^{\pm} = a\left(1 + \frac{R_4}{2} \pm \frac{D_4}{2}\right) , \tag{2.16a}$$

$$r_2^{\pm} = a\left(1 - \frac{R_4}{2} \pm \frac{E_4}{2}\right) , \tag{2.16b}$$

where

$$R_4^2 = 6 + y_3 , \tag{2.16c}$$

$$D_4^2 = -R_4^2 + 12 + \frac{16}{R_4} , \tag{2.16d}$$

$$E_4^2 = -R_4^2 + 12 - \frac{16}{R_4} . \tag{2.16e}$$

The minimum value y_3 can take is -6, so $R_4^2 \geq 0$. Since $E_4^2 \leq 0$, the roots $r_2^\pm$ are complex and do not concern us here. If we now demand that $D_4^2 \geq 0$, we again find the constraint (2.14). So, if equation (2.14) is satisfied, the two roots $r_1^\pm$ are real for all z, and for $z < 1$, the roots are real for all λ.

The behavior of the roots $r_1^\pm$ as function of z is shown in Figure 1. For z going from 0 to 1, the branch r_1^+ runs from $-\infty$ to a maximum value, $r_{\rm E} \leq 0$, and decreases again to $-\infty$. Since this part of branch r_1^+, which we will denote by $r_{1,a}^+$, is negative, it describes the stationary part of super-Eddington ($L > L_{\rm E}$) flow. We will show later that this type of flow is supersonic everywhere. There is no stationary subsonic part, which matches the supersonic part to the flow near the star. For z going from 1 to ∞, the branch r_1^+ runs from ∞ to 1. This part of branch r_1^+, henceforth $r_{1,b}^+$, describes subsonic flow. For $\lambda^2 > \lambda_{\rm max}^2$, part of this branch becomes complex, namely between z_1 and z_2, which are the two roots of $\lambda^2 = \frac{1}{2}[3(\gamma - 1)/4]^3[z^{2/(\gamma-1)}/(z-1)^3]$. These solutions are relevant for shocked flow. For z going from 0 to ∞, r_1^- runs from ∞ to 0. This branch describes supersonic accretion or outflow, with sound speed, and thus also temperature, decreasing away from the star. We now discuss the different branches in more detail.

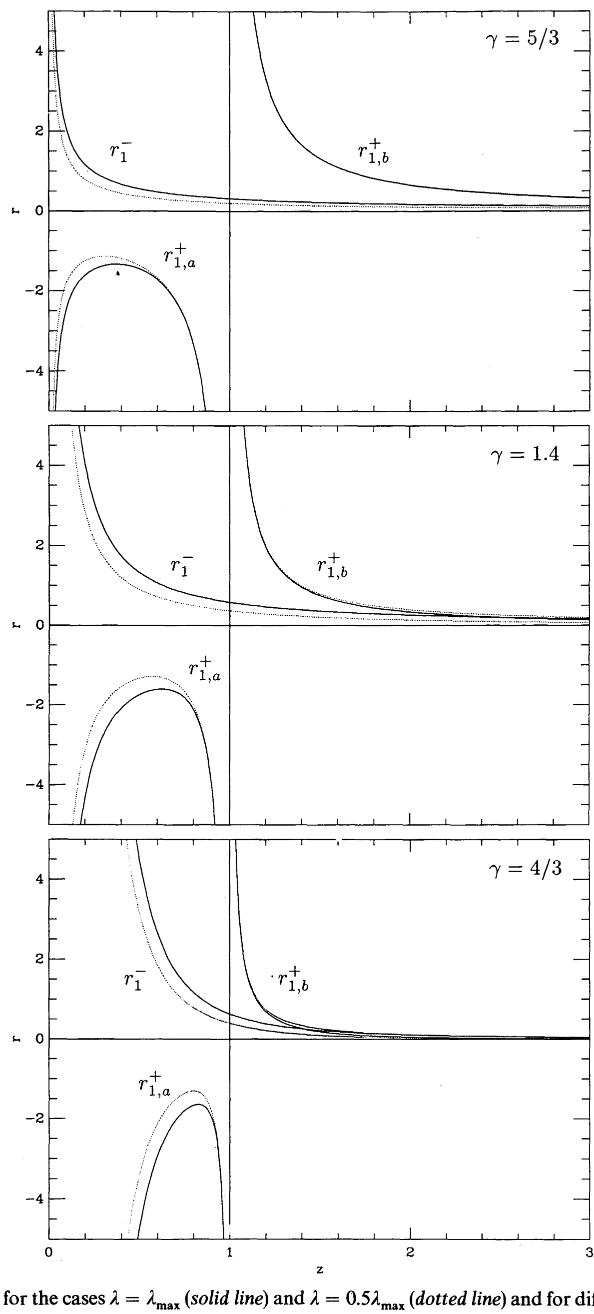

FIG. 1.—Two real branches of eq. (2.9) for the cases $\lambda = \lambda_{\rm max}$ (*solid line*) and $\lambda = 0.5\lambda_{\rm max}$ (*dotted line*) and for different values of γ as indicated on the plots

2.1.2. *Flow Types*

2.1.2.1. Super-Eddington flow ($L > L_E$)

Evaluating the differential (2.11b) at constant λ one finds immediately that the maximum of the branch $r_{1,a}^{+}$ in Figure 1 corresponds to $M = 1$. Denoting the coordinates of this point by (z_E, r_E) it follows from equation (2.10d) that

$$r_E = -\left(\frac{\lambda^2}{z_E^{(\gamma+1)/(\gamma-1)}}\right)^{1/4} . \tag{2.17}$$

For given λ and γ, equations (2.9) and (2.17) determine the values r_E and z_E. If λ decreases, the point r_E moves closer to the star. From equations (2.17) and (2.10d) and Figure 1 it is easily seen that the branch $r_{1,a}^{+}$ is subsonic (supersonic) for $z > z_E$ ($z < z_E$). The subsonic part has temperature increasing away from the star, which does not seem physically acceptable. Therefore, stationary super-Eddington flow will be supersonic. For all values of λ it is described by the branch $r_{1,a}^{+}$ in the interval $0 < z < z_E$.

2.1.2.2. Sub-Eddington Flow ($L < L_E$)

We will first show that, for $\lambda^2 \leq \lambda_{max}^2$, the branch $r_{1,b}^{+}$ describes subsonic flow and r_1^{-} describes supersonic flow. Again from equation (2.11b) at constant λ we find that $M(r)$ reaches an extremum at a distance $r = r_s$ which is independent of λ:

$$r_s = \frac{5 - 3\gamma}{4} . \tag{2.18}$$

If we now consider equations (2.11a) and (2.11b) at constant r and vary M^2 from 0 to ∞, we find that $\lambda^2(M^2)$ rises monotonously from 0 to a maximum at $M = 1$ (the maximum being the value in equation [2.14] if $r = r_s$) and then decreases montonously to 0. Let us consider the points corresponding to any chosen r on the branches $r_{1,b}^{+}$ and r_1^{-} in Figure 1. Denote the corresponding z-values and Mach numbers by $z_{1,b}^{+}$, $M_{1,b}^{+}$, z_1^{-} and M_1^{-}, respectively. Since $z_1^{-} < z_{1,b}^{+}$ it follows from equation (2.10d) that $M_1^{-} > M_{1,b}^{+}$; on the other hand these Mach numbers are also the roots of equation (2.11a) for the given values of λ and r. Since these roots are always separated by the value $M = 1$, we conclude that $M_1^{-} > 1$ and $M_{1,b}^{+} < 1$ for all r and $\lambda < \lambda_{max}$. Only if $\lambda = \lambda_{max}$ and $r = r_s$, the roots coincide.

If $\lambda < \lambda_{max}$, a continuous flow starting with $v < c$ at the stellar surface or at infinity is bound to remain along the subsonic branch $r_{1,b}^{+}$; Parker (1963) pointed out that outflow in this case could only occur if there were an unrealistically high inward pressure acting upon the wind at some distance from the star. Therefore the branch $r_{1,b}^{+}$ with $\lambda < \lambda_{max}$ is only useful to describe subsonic accretion (Bondi's 1952 Type II solution); it is characterized by the fact that, from $r = r_s$ toward the star, due to compression of the gas, the sound speed increases faster than the flow velocity so that the flow remains subsonic even though the flow velocity v may tend to ∞ (which is the case if $\gamma > 3/2$; cf. eq. [2.22c]).

If $\lambda = \lambda_{max}$ then both inflow and outflow can switch in a continuous way from the branch $r_{1,b}^{+}$ to r_1^{-} at the sonic point r_s; if they do not switch, the derivatives of the flow variables become discontinuous at this point (see also § 2.1.4). The transonic solutions describe respectively stellar wind (which is subsonic at the stellar surface) and Bondi accretion (which is subsonic far away from the star).

The flow profiles $\rho(r)$, $v(r)$, $c(r)$, and $M(r)$ for these types of flow are illustrated in Figure 2 for the case $\gamma = 1.4$. The right half of Figure 2*b* is essentially the same as Parker's (1963) Figure 5.3 (apart from the actual value of the flow parameters and from the units we use). We do not show the branches corresponding to $\lambda > \lambda_{max}$. The left half of Figure 2*b* is irrelevant for the case of solar wind, so it was not considered by Parker.

2.1.3. *Asymptotic Expansions*

We now derive asymptotic expansions for the solutions by examining the behavior of the roots (eq. [2.16]) for $z \to 0$, $z \to 1$, $z \to \infty$, and $z \to z_E$: equation (2.16) yields the exponent for the asymptotic power-law behavior of $r(z)$, while from equation (2.10) we then find the behavior of the other flow variables. The results can be summarized as follows:

1. In supersonic flow, $r \to \infty$ for $z \to 1$ and $r \to 0$ for $z \to \infty$;
2. Along the subsonic branch, $r \to \infty$ for $z \to 1$ and $r \to 0$ for $z \to \infty$;
3. Finally in super-Eddington flow, $r \to \infty$ for $z \to 0$ and $r \to r_E$ for $z \to z_E$.

2.1.3.1. Super-Eddington Flow

For the branch $r_{1,a}^{+}$, we find, close to the base of the flow (at r_E):

$$r \to r_E\left[1 + G\,\frac{(z - z_E)^2}{z_E^2}\right] \to r_E , \tag{2.19a}$$

$$\rho \to z_E^{1/(\gamma-1)}\left(1 \pm \frac{1}{\gamma - 1}\sqrt{\frac{r - r_E}{r_E G}}\right) \to z_E^{1/(\gamma-1)} , \tag{2.19b}$$

$$v \to \sqrt{z_E}\left(1 \mp \frac{1}{\gamma - 1}\sqrt{\frac{r - r_E}{r_E G}}\right) \to \sqrt{z_E} , \tag{2.19c}$$

$$c \to \sqrt{z_E}\left(1 \pm \frac{1}{2}\sqrt{\frac{r - r_E}{r_E G}}\right) \to \sqrt{z_E} , \tag{2.19d}$$

$$M \to 1 \mp \frac{\gamma + 1}{2(\gamma - 1)} \sqrt{\frac{r - r_E}{r_E G}} \to 1 \,, \tag{2.19e}$$

$$G \equiv \frac{1}{2} \frac{\gamma + 1}{\gamma - 1} \frac{z_E r_E}{4(z_E - 1) r_E - 3(\gamma - 1)} \,. \tag{2.19f}$$

The branch with the lower sign in equations (2.19) corresponds to the super-Eddington flow there. z_E and r_E are the solutions of

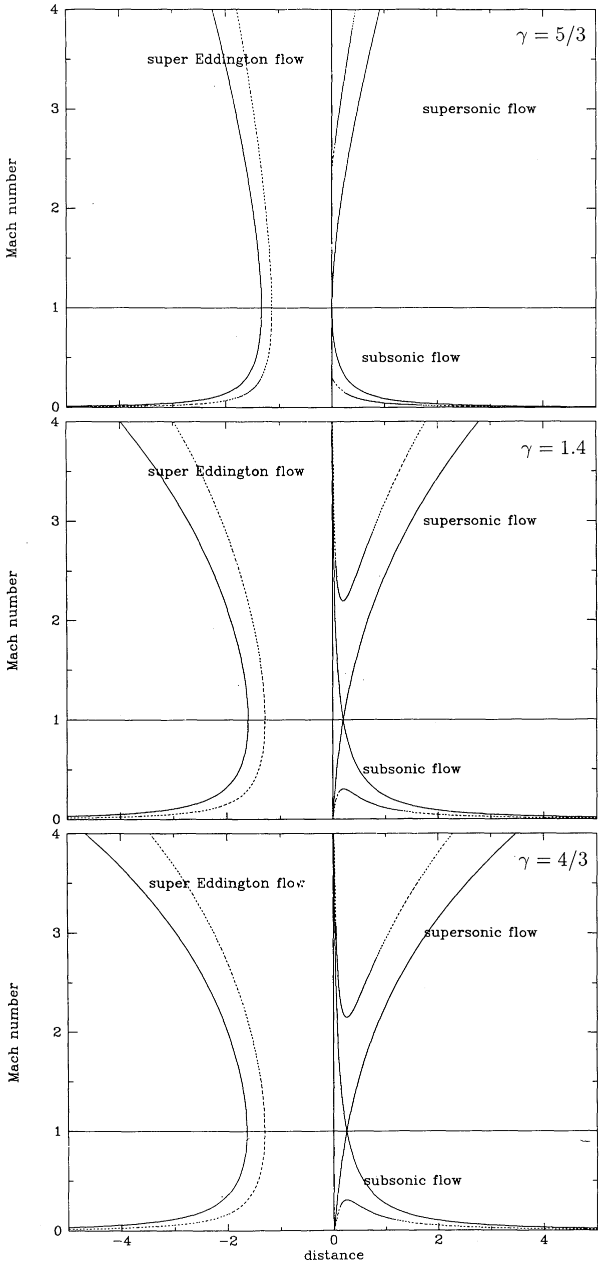

FIG. 2a

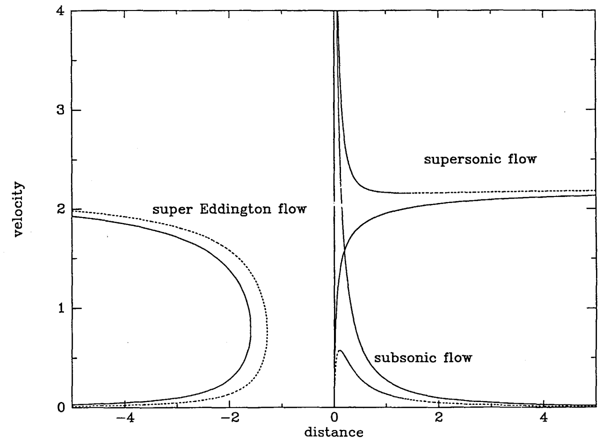

FIG. 2b

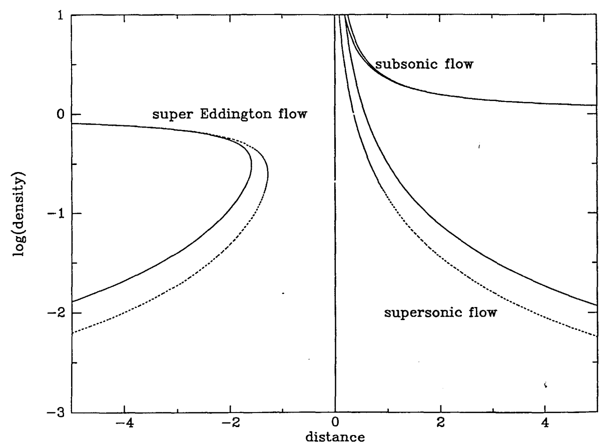

FIG. 2c

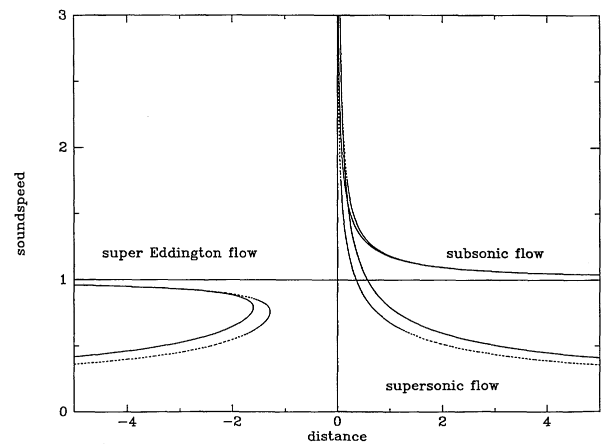

FIG. 2d

FIG. 2.—Flow profiles for (*a*) Mach number (for different γ-values as indicated on the plots), (*b*) velocity, (*c*) density, and (*d*) sound speed; solid lines correspond to $\lambda = \lambda_{max}$ and dashed lines to $\lambda = 0.5\lambda_{max}$. In (*b*)–(*d*) we considered only $\gamma = 1.4$ ($\lambda_{max} = 0.625$). The three different types of flow (supersonic, subsonic, and super-Eddington flow) are indicated.

equations (2.9) and (2.17). When we move away from the star, for $z \to 0$, the results for this branch read:

$$r \to -\frac{F}{z^{1/2(\gamma-1)}} - \frac{\gamma-1}{4} \to -\infty , \tag{2.20a}$$

$$\rho \to \left(\frac{F}{-r}\right)^2 \left(1 - \frac{\gamma-1}{2r}\right) \to 0 , \tag{2.20b}$$

$$v \to \frac{\lambda}{F^2}\left(1 + \frac{\gamma-1}{2r}\right) \to \sqrt{\frac{2}{\gamma-1}} , \tag{2.20c}$$

$$c \to \left(\frac{F}{-r}\right)^{\gamma-1}\left[1 - \frac{(\gamma-1)^2}{4r}\right] \to 0 , \tag{2.20d}$$

$$M \to \frac{\lambda(-r)^{\gamma-1}}{F^{\gamma+1}}\left[1 + \frac{(\gamma-1)(\gamma+1)}{4r}\right] \to \infty , \tag{2.20e}$$

$$F \equiv \left[\frac{(\gamma-1)\lambda^2}{2}\right]^{1/4} . \tag{2.20f}$$

Equations (2.20a)–(2.20f) are valid for $\gamma > 3/2$. For smaller values of γ, other terms in equation (2.20a) appear that diverge for $z \to 0$, but they remain small with respect to the dominant term given in equation (2.20).

2.1.3.2. Subsonic Accretion

For the branch $r^{+}_{1,b}$, corresponding to subsonic accretion, we find, near $z = 1$:

$$r \to \frac{\gamma-1}{z-1} \to \infty , \tag{2.21a}$$

$$\rho \to 1 + \frac{1}{r} \to 1 , \tag{2.21b}$$

$$v \to \frac{\lambda}{r^2} \to 0 , \tag{2.21c}$$

$$c \to 1 + \frac{\gamma-1}{2r} \to 1 , \tag{2.21d}$$

$$M \to \frac{\lambda}{r^2} \to 0 , \tag{2.21e}$$

which describes accretion flow at infinity. Upon approach of the accreting object, pressure gradients will start building up, slowing down the infalling gas. In the limit $z \to \infty$, we find

$$r \to \frac{\gamma-1}{z} \to 0 , \tag{2.22a}$$

$$\rho \to \left(\frac{\gamma-1}{r}\right)^{1/(\gamma-1)} \to \infty , \tag{2.22b}$$

and

$$v \to \frac{\lambda}{(\gamma-1)^{1/(\gamma-1)}} \frac{1}{r^{(2\gamma-3)/(\gamma-1)}} \begin{cases} \to 0 & \text{for } \gamma < 3/2 \\ \to 4\lambda[1 + 8(\lambda^2 r)^{2/3}] & \text{for } \gamma = 3/2 \\ \to \infty & \text{for } \gamma > 3/2 \end{cases} , \tag{2.22c}$$

$$c \to \sqrt{\frac{\gamma-1}{r}} \to \infty , \tag{2.22d}$$

$$M \to \frac{\lambda}{(\gamma-1)^{(\gamma+1)/[2(\gamma-1)]}} r^{(5-3\gamma)/[2(\gamma-1)]} \to 0 . \tag{2.22e}$$

Equation (2.22) shows that the maximum infall velocity occurs for $\gamma = 5/3$, in which case $v \sim r^{-1/2}$, as in the free-fall case. For $\gamma = 3/2$, the flow velocity approaches a constant, finite value. For $\gamma < 3/2$, the flow velocity goes to zero. The lower infall velocities for smaller γ are compensated by a more rapid divergence of the density. Remark that for $\gamma \geq 3/2$ the flow *cannot* be matched smoothly onto any surface, a fact which is well known in the literature on stellar winds (Parker 1963), but seems to have been overlooked in several studies of accretion.

2.1.3.3. *Supersonic Branch*

For the supersonic branch, we find, close to the star ($z \to \infty$):

$$r \to \frac{r_A}{z^{2/[3(\gamma-1)]}} \to 0 \, , \tag{2.23a}$$

$$\rho \to \left(\frac{r_A}{r}\right)^{3/2} \to \infty \, , \tag{2.23b}$$

$$v \to \sqrt{\frac{2}{r}} \to \infty \, , \tag{2.23c}$$

$$c \to \left(\frac{r_A}{r}\right)^{3(\gamma-1)/4} \to \infty \, , \tag{2.23d}$$

$$M \to \sqrt{\frac{2}{r_A}} \left(\frac{r_A}{r}\right)^{(5-3\gamma)/4} \to \infty \, , \tag{2.23e}$$

$$r_A \equiv \left(\frac{\lambda^2}{2}\right)^{1/3} \, . \tag{2.23f}$$

In the case of accretion, pressure support now becomes unimportant close to the star, leading to a divergence of the Mach number of the flow; equation (2.23c) shows that the velocity field approaches free-fall $v \sim r^{-1/2}$. For the asymptotic form of the supersonic wind we again find the form (2.20) (with $r \to -r$). The velocity approaches a constant value, and so $\rho \sim r^{-2}$; the Mach number of the flow tends to infinity.

We illustrate the asymptotic behavior of these profiles in Figure 3.

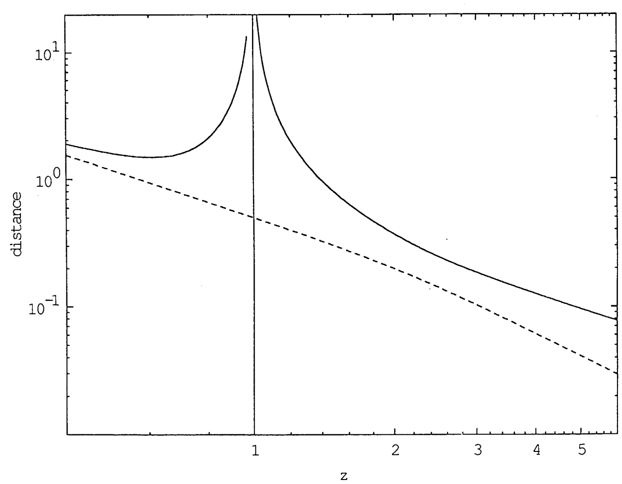

FIG. 3*a*

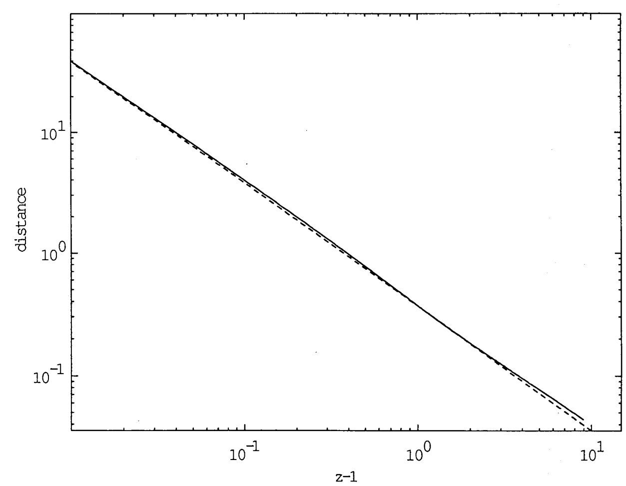

FIG. 3*b*

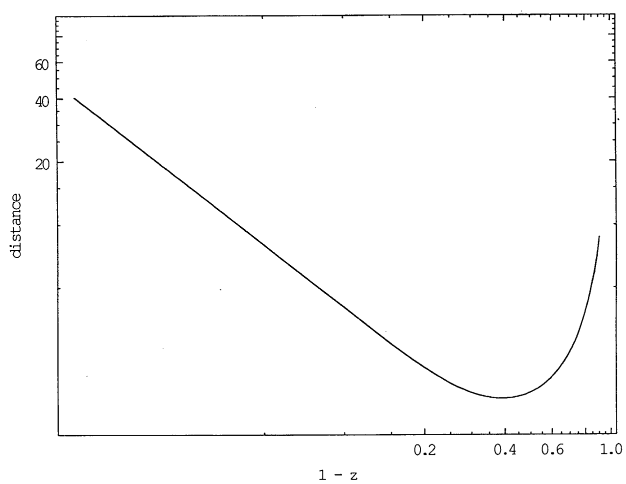

FIG. 3*c*

FIG. 3.—Asymptotic behavior of the absolute value of the roots $r_1^{\pm}$ (eq. [2.16]) of the quartic equation for the case $\gamma = 1.4$, $\lambda = 0.5 < \lambda_{max}$. (*a*) We show in particular the behavior for $z \to 0$ and $z \to \infty$ (z is square of the sound speed). Solid line: the branch r_1^+; dashed line: the branch r_1^-. (*b*) Same as (*a*) for the super-Eddington flow near $z \to 1$. (*c*) Same as (*a*) for subsonic accretion near $z \to 1$. Dashed line is a straight line with slope -1, added for comparison.

2.1.4. *The Sonic Point*

When the accretion rate is maximal, $\lambda = \lambda_{max}$, equation (2.14), the flow can change from the supersonic to the subsonic branch in the sonic point r_s. We will derive the form of the flow profile close to r_s. In this *transonic* case, all flow variables, as well as their (spatial) derivatives are continuous. The location of the (well-known) sonic point is given by equation (2.18). Note that the sonic point is in the origin, for the case $\gamma = 5/3$. Transforming back to physical variables, for example, in the context of accretion of gas onto a protostar in a molecular cloud, we find:

$$r_s = \frac{5-3\gamma}{4\gamma}\, 1.4 \times 10^4 \text{ AU}\, \frac{m/M_\odot\, \mu/(2\mu_H)}{T_\infty/10 \text{ K}}, \tag{2.24}$$

where m is the protostellar mass (corrected for radiation pressure), μ is the mean molecular weight, and T_∞ is the temperature at ∞. In the late stages of cloud collapse, a central core mass forms and subsequently grows by accretion. This stage of cloud collapse is represented well by Bondi-type accretion (see, e.g., Boss & Black 1982; Zinnecker & Tscharnuter 1983). Equation (2.24) shows that unless γ is close to 5/3 a large part of the flow profile will in fact be supersonic. In the case of accretion onto a black hole, powering an AGN, we find:

$$r_s = \frac{5-3\gamma}{4\gamma}\, 7.4 \times 10^2 r_g\, \frac{\mu/\mu_H}{T_\infty/10^6 \text{ K}}, \tag{2.25}$$

where r_g is the gravitational radius of the hole. For the (sub-Eddington) outflow case, we find

$$r_s = \frac{5-3\gamma}{\gamma-1}\, 0.59\, R_\odot\, \frac{m/M_\odot}{(v_\infty/400 \text{ km s}^{-2})}, \tag{2.26}$$

where v_∞ is the terminal flow speed.

Finally, we give the expansions in the vicinity of the sonic point

$$r = r_s\left[1 - \frac{4}{3} r_s \frac{z - z_s}{\gamma - 1}\left(1 \mp \frac{z - z_s}{|z - z_z|}\sqrt{\frac{r_s}{2}}\right)\right], \tag{2.27a}$$

$$\rho = \left(\frac{1}{2r_s}\right)^{1/(\gamma-1)}\left[1 + \frac{4}{\gamma+1}(r_s - r)\left(1 \pm \sigma\sqrt{\frac{r_s}{2}}\right)\right], \tag{2.27b}$$

$$v = \frac{1}{\sqrt{2r_s}}\left[1 + \frac{4}{\gamma+1}\frac{r_s - r}{r_s}\left(\frac{\gamma-1}{2} \mp \sigma\sqrt{\frac{r_s}{2}}\right)\right], \tag{2.27c}$$

$$c = \frac{1}{\sqrt{2r_s}}\left[1 + 2\frac{\gamma-1}{\gamma+1}\frac{r_s - r}{r_s}\left(1 \pm \sigma\sqrt{\frac{r_s}{2}}\right)\right], \tag{2.27d}$$

$$M = 1 \mp \sqrt{\frac{2}{r_s}}|r_s - r|, \tag{2.27e}$$

where we have used $z_s \equiv 1/2r_s$ and $\sigma \equiv r_s - r/|r_s - r|$. The upper (lower) sign denotes the sub- (super-) sonic branch. For the transonic *accretion* flow, this gives for the derivatives in r_s:

$$\left.\frac{\partial \rho}{\partial r}\right|_{r=r_s} = -\frac{4}{\gamma+1}\left(\frac{1}{2r_s}\right)^{1/(\gamma-1)}\left(1 - \sqrt{\frac{r_s}{2}}\right), \tag{2.28a}$$

$$\left.\frac{\partial v}{\partial r}\right|_{r=r_s} = \frac{-4}{\gamma+1}\frac{1}{\sqrt{2r_s^3}}\left(\frac{\gamma-1}{2} + \sqrt{\frac{r_s}{2}}\right), \tag{2.28b}$$

$$\left.\frac{\partial c}{\partial r}\right|_{r=r_s} = -2\frac{\gamma-1}{\gamma+1}\frac{1}{\sqrt{2r_s^3}}\left(1 - \sqrt{\frac{r_s}{2}}\right), \tag{2.28c}$$

$$\left.\frac{\partial M}{\partial r}\right|_{r=r_s} = -\sqrt{\frac{2}{r_s}}, \tag{2.28d}$$

and for outflow, one has to substitute $\surd \to -\surd$. Note that for the transonic solutions, all flow variables and their derivatives are continuous in the sonic point.

2.2. *Shocked Flow*

We now consider a stationary adiabatic shock, linking a supersonic branch to a subsonic one. Shocks in accretion- or wind-type flows have been considered in many earlier studies (e.g., Parker 1963; Shapiro & Salpeter 1975; Bailyn et al. 1985), mostly between regions with altogether different characteristics (e.g., differing in heating or cooling rates, degree of ionization, flow pattern, etc.). Presently we are interested in shocks between regions which are as much alike as possible, that is, having the same value for the polytropic exponent and with a flow regime satisfying equation (2.9). First we show that such a shock may occur at any position r_{sh} in the supersonic region; the flow profile behind the shock is then determined by r_{sh}. Conversely, if the flow profile behind the shock is determined otherwise (e.g., by a boundary condition downstream), then the position of the shock is fixed, and we indicate how it can be calculated. We will also show that the shock connects a region with $\lambda \leq \lambda_{max}$ with one where $\lambda > \lambda_{max}$.

Conventionally we denote by (1) and (2), respectively, the supersonic and the subsonic regions. Accordingly, the values of the flow variables ρ, v, c, p *in physical units* at the shock, as well as the parameters c_*, ρ_*, λ, and κ characterizing the flow in each region, will be given a subscript 1 or 2.

The jump conditions at an adiabatic shock are

$$\rho_1 v_1 = \rho_2 v_2 \,, \tag{2.29a}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \,, \tag{2.29b}$$

$$c_1^2 + \frac{\gamma - 1}{2} v_1^2 = c_2^2 + \frac{\gamma - 1}{2} v_2^2 \,. \tag{2.29c}$$

Here the flow velocities with respect to the shock, v_1 and v_2, are also the velocities with respect to the star since we consider a standing shock. If we put

$$\frac{\rho_2}{\rho_1} = t \,, \tag{2.30}$$

the conditions (2.29) lead to

$$t = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \,; \quad M_1 = v_1/c_1 \tag{2.31a}$$

and

$$\frac{c_2^2}{c_1^2} = 1 + \frac{\gamma - 1}{2} M_1^2\left(1 - \frac{1}{t^2}\right) . \tag{2.31b}$$

Since γ is conserved through the shock

$$\frac{p_2}{p_1} = \frac{\rho_2 c_2^2}{\rho_1 c_1^2} \,. \tag{2.31c}$$

It follows from equation (2.31c) that the constant in Bernoulli's equation must be the same for both regions, so that the unit of velocity:

$$c_{*1} = c_{*2} = c_* \tag{2.32}$$

is also the same at both sides of the shock. On the other hand it follows from equations (2.31) and (2.32) that

$$\frac{\kappa_2}{\kappa_1} = \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \frac{c_2^2}{c_1^2}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma - 1} = \left[1 + \frac{\gamma - 1}{2} M_1^2\left(1 - \frac{1}{t^2}\right)\right]t^{1-\gamma} \tag{2.33}$$

does not equal one so that according to equation (2.6) the respective units of density must be different; one has in fact

$$\frac{\rho_{*2}}{\rho_{*1}} = \left(\frac{\kappa_1}{\kappa_2}\right)^{1/(\gamma-1)} = \frac{\rho_2}{\rho_1}\left(\frac{c_1^2}{c_2^2}\right)^{1/(\gamma-1)} \tag{2.34}$$

and finally, since the flow rate is conserved through the shock (cf. eq. [2.29a]) and taking account of equations (2.8) and (2.32):

$$\frac{\lambda_2}{\lambda_1} = \frac{\rho_{*1}}{\rho_{*2}} = \frac{\rho_1}{\rho_2}\left(\frac{c_2^2}{c_1^2}\right)^{1/(\gamma-1)} = \frac{1}{t}\left[1 + \frac{\gamma - 1}{2} M_1^2\left(1 - \frac{1}{t^2}\right)\right]^{1/(\gamma-1)} . \tag{2.35}$$

Now we require that equation (2.9) is satisfied both before and after the shock; to this end it is sufficient to require that it is satisfied by the values of both sets of flow variables at the shock itself. Because of equations (2.8) and (2.32), the dimensionless distance of the shock, r_{sh}, is the same at both sides while the respective dimensionless sound speeds are related by equation (2.31b). Multiplying equation (2.9) by $(z - 1)/(\gamma - 1)$ and subtracting the resulting expressions for either side of the shock, we obtain

$$\left(\frac{z_2 - z_1}{\gamma - 1}\right)r_{sh}^4 = \frac{1}{2}\lambda_1^2 z_1^{-2/(\gamma-1)}\left[1 - \left(\frac{\lambda_2}{\lambda_1}\right)^2\left(\frac{z_2}{z_1}\right)^{-2/(\gamma-1)}\right] = \frac{1}{2}\lambda_1^2 z_1^{-2/(\gamma-1)}\left[1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right] , \tag{2.26}$$

and finally, using equations (2.30) and (2.31)

$$M_1^2 r_{sh}^4 = \lambda_1^2 z_1^{-(\gamma+1)/(\gamma-1)} \,. \tag{2.37}$$

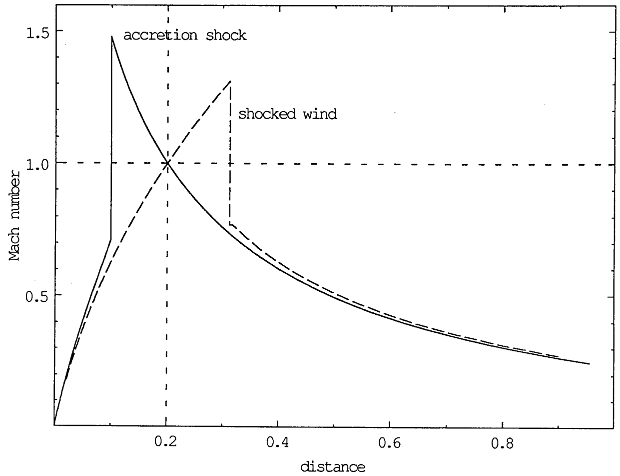

FIG. 4.—Mach number vs. distance to the star for two shocked flows, for the case $\gamma = 1.4$. Solid line represents accretion flow. Value of the Mach number at the position of the shock is 1.48. The part of the profile between the shock, and the star has an accretion rate of $\lambda = 0.674 > \lambda_{max}$. Dashed line represents shocked wind. Here the value of the Mach number at the position of the shock is 1.31. The part of the flow beyond the shock has $\lambda = 0.639 > \lambda_{max}$.

It follows from equation (2.10d) that this equation is identically satisfied, so we conclude that steady supersonic flow may always be linked, through an adiabatic shock, to subsonic flow with the same value for the polytropic exponent. The shock may occur at any distance $r_{sh} < r_s$ in the case of accretion, any distance $r_{sh} > r_s$ in the case of ordinary stellar wind, and any distance $r_{sh} > r_E$ in the case of super-Eddington wind. Figure 4 shows an example of a shock in an accretion profile, and in a wind. Notice that equation (2.35) implies $\lambda_2 \geq \lambda_1 = \lambda_{max}$, so in the case of sub-Eddington flow, the subsonic branch to which the supersonic one is linked, is of the type which is forbidden for *continuous* steady flow.

Suppose now that the subsonic flow profile is given; then λ_1/λ_2 is known and equation (2.35) can be seen as an equation for the Mach number M_1, that is, after substitution of equation (2.31a):

$$\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2} \left[1 + \frac{2(\gamma - 1)}{(\gamma + 1)2M_1^2} (M_1^2 - 1)(\gamma M_1^2 + 1) \right]^{1/(\gamma - 1)} = \frac{\lambda_2}{\lambda_1} . \tag{2.38}$$

The location of the shock is obtained by solving M_1^2 from equation (2.38) and substituting it into equation (2.11a).

3. STABILITY

3.1. *Perturbation Analysis*

We study the stability of spherical polytropic steady flow by considering the behavior of a perturbation on the flow rate, $f_1 = r^2(\rho_0 v_1 + \rho_1 v_0)$, rather than perturbations ρ_1, v_1, p_1 on the several flow variables.

We limit ourselves to an "Eulerian" view of stability, that is, we do *not* consider the possibility that an originally linear perturbation grows, due to propagation in a nonuniform medium, to such an extent that the linear treatment breaks down (see, e.g., Feix 1963). For example, Parker (1966) showed that the amplitude of a sound wave, propagating outward in the solar wind, grows like $\sim 1/(vc)^{1/2}$. This might lead to nonlinear behavior, but is nevertheless characterized by a completely *real* frequency and is therefore generally not considered to be an actual instability.

PSO derived a linearized partial differential equation for f_1 in a study of the stability of Bondi accretion (and of course this equation applies to outflow as well); we rewrite it as follows:

$$\frac{1}{v} \frac{\partial}{\partial r} \left[v(v^2 - c^2) \frac{\partial f_1}{\partial r} \right] + 2 \frac{\partial}{\partial r} \left(v \frac{\partial f_1}{\partial t} \right) + \frac{\partial^2 f_1}{\partial^2 t} = 0 . \tag{3.1a}$$

For a general discussion of stability it is convenient to substitute a complex form for the solution, $f_1 = h(r)e^{i\omega t}$, which leads to

$$\frac{1}{v} [v(v^2 - c^2)h']' + 2i\omega(vh)' - \omega^2 h = 0 , \tag{3.1b}$$

where the prime denotes derivation with respect to r. Multiplying this equation by vh^* and separating the real and imaginary parts, we get, respectively,

$$[v(v^2 - c^2)(|h|^2)']' - v(v^2 - c^2)|h'|^2 - 2\omega v^2 \text{ Im } (h'h^*) - \omega^2 v|h|^2 = 0 \tag{3.2a}$$

and

$$[v(v^2 - c^2) \text{ Im } (h'h^*)]' + \omega(v^2|h|^2)' = 0 , \tag{3.2b}$$

which leads to a constant of the motion

$$v(v^2 - c^2) \text{ Im } (h'h^*) + \omega v^2 |h|^2 = C . \tag{3.3}$$

The constant will often be zero in practice (e.g., for accretion from an infinite medium where $v_\infty = 0$). We shall see later that its value will mostly be irrelevant in a discussion of stability. Substituting Im $(h'h^*)$ from equation (3.3) into equation (3.2) and integrating over an arbitrary interval $[r_1, r_2]$ (not containing a singular point of the equation or a discontinuity of the flow), we get

$$A\omega^2 - 2B\omega + C = 0 , \tag{3.4a}$$

where

$$A = \int_{r_1}^{r_2} v\left(\frac{v^2 + c^2}{v^2 - c^2}\right)|h|^2\, dr , \tag{3.4b}$$

$$B = C \int_{r_1}^{r_2} \frac{v}{v^2 - c^2}\, dr , \tag{3.4c}$$

$$C = \tfrac{1}{2}[v(v^2 - c^2)(|h|^2)']_{r_1}^{r_2} - \int_{r_1}^{r_2} v(v^2 - c^2)|h'|^2\, dr . \tag{3.4d}$$

In order to assess the stability of the flow in a given interval $[r_1, r_2]$ we may consider equation (3.4a) as a pseudoquadratic equation for ω which allows real values for ω only if its discriminant $D = B^2 - AC$ is positive. One sees immediately that the sign of D will *not* depend on the signs of v and $v^2 - c^2$, so that no distinction must be made between cases of accretion and outflow, nor between intervals in which the flow is either purely subsonic or purely supersonic. Upon closer inspection one finds that only the term

$$Q = -\tfrac{1}{2}A[v(v^2 - c^2)(|h|^2)']_{r_1}^{r_2} = -\tfrac{1}{2}|A|[|v(v^2 - c^2)|(|h|^2)']_{r_1}^{r_2} \tag{3.5}$$

could be a negative contribution to D.

Clearly one cannot discuss the stability of the flow in a given interval without knowing (or making assumptions on) the boundary conditions. If these are such that Q is negative, the discriminant D itself may yet be positive. However, the value of Q depends only on the boundary conditions, whereas the value of D depends on the behavior of the solution over the whole interval $[r_1, r_2]$, so the sign of D would have to be checked for every possible solution of equation (3.1) individually, and one might find that some of the solutions will grow exponentially in time while others (satisfying the same boundary conditions) remain bound. Anyway, with such boundary conditions it would be impossible to state that the flow is stable with respect to any perturbation, so we conclude that, for practical purposes,

$$Q \geq 0 \tag{3.6}$$

can be used as a necessary and sufficient condition for " stability of the flow " in the interval considered.

The perturbation analysis thus yields a clear criterion to assess the stability of a given flow regime. While as yet the available astrophysical data are probably insufficient for a successful application of this criterion, we believe that it may be particularly useful in numerical simulations, where one *imposes* (static) boundary conditions and where it should be helpful to know beforehand whether the conditions imposed should lead to a stable flow regime.

The foregoing results were obtained assuming only that the interval considered does not contain a sonic point or a shock. As far as stability is concerned a purely supersonic region is not very interesting since, as Garlick (1979) pointed out, any disturbance is carried away in a finite time so that no instabilities can grow there. In the following sections we concentrate on subsonic, transonic, and shocked flow, respectively.

3.2. *Subsonic Flow*

As an example of the application of the criterion (3.6) we reconsider some of the situations discussed by PSO:

1. Subsonic accretion flow perturbed by standing waves: if there are two radii r_1 and r_2 at which the perturbation vanishes at all times, obviously $Q = 0$ in equation (3.5), and according to equation (3.6) the flow between r_1 and r_2 is stable, in agreement with the conclusion of PSO; however, we also find from equation (3.5) that the same conclusion will hold if the perturbation does not vanish but its derivative does, at either or both of the points considered.

2. Subsonic accretion flow in an infinite medium ($v_\infty = 0$) perturbed by radially traveling waves: using a WKB approximation PSO showed (without using boundary conditions) that accretion flow is stable with respect to high-frequency waves (although it should be noted that in doing so they used $\lim_{r \to 0} (1/vc)^{1/2} \to 0$ which is true only for $\gamma > 1.4$ since $(vc)^{1/2} \sim r^{(7-5\gamma)/[4(\gamma-1)]}$ (cf. eq. [2.22]). Within the framework of the present approach we put $r_2 = \infty$ snd take for r_1 the radius of the star or of some buffer layer between the flow and the star; then

$$Q = \tfrac{1}{2}|A||v(v^2 - c^2)|(|h|^2)'|_{r_1} ,$$

which shows that (if the effect of the boundary layer on a disturbance can at all be represented by a static boundary condition) stability is *ensured* only if the boundary conditions make $(|h|^2)' \|_{r_1} \geq 0$. There is a very plausible argument to this effect, notably that $|h|' < 0$ would mean that the disturbance is actually enhanced by the boundary layer (which seems improbable). But no actual proof can be given until the physics of the boundary layer are properly known. On the other hand, using equations (3.4) and (3.5) we can understand why stability with respect to high-frequency perturbation could be demonstrated without reference to the boundary conditions: since $B = 0$ we have $\omega = (-A/C)^{1/2}$; in a given flow regime higher frequencies will correspond to shorter wavelengths, that is, to higher average values for $|h'|^2$ over the whole interval; this implies that for disturbances with sufficiently high frequencies the second term in equation (3.4d) will dominate so that ω will be real.

A more detailed study of the behavior of disturbances in spherical polytropic flow evidently requires some knowledge of the general solution of the hyperbolic partial differential equation (p.d.e.) (3.1a). We now discuss the classical procedure to obtain this solution, mainly in order to point out the problems it entails and to provide a possible basis for further work. It is easily seen that the variables (r, t) in equation (3.1a) are not separable. However, looking at the equation from the physical point of view one would expect the solutions to look like modulated traveling waves with a phase velocity which is position-dependent since they travel through a nonuniform medium, that is,

$$f_1(r, t) = g(r; \omega) \cos \omega[t - \tau(r)] . \tag{3.7}$$

If this form is correct, then a change of variables $(r, t) \rightarrow (r, s)$, where s is defined by

$$s = t - \tau(r) , \tag{3.8a}$$

could make the p.d.e. separable. If we substitute equation (3.8a) and do the replacements

$$\frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial r} - \tau' \frac{\partial}{\partial s} , \qquad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial s} \tag{3.8b}$$

in equation (3.1a) and require that the terms containing $\partial^2/\partial r\, \partial s$ shoud vanish, we find that the phase function $\tau(r)$ must satisfy

$$\tau'(r) = \frac{v}{v^2 - c^2} \tag{3.9a}$$

or

$$\tau = \int^r \frac{v\, dr}{v^2 - c^2} = \frac{1}{2} \int^r \frac{dr}{v + c} + \frac{1}{2} \int^r \frac{dr}{v - c} , \tag{3.9b}$$

where the lower integration boundary is left out to indicate that this equation determines τ only up to an integration constant. The last equality in equation (3.9b) allows an interesting interpretation: the characteristics of equation (3.1a) are given by

$$\tau_{\pm} = \int^r \frac{dr}{v \pm c} \tag{3.9c}$$

so that $\tau = \frac{1}{2}(\tau_+ + \tau_-)$; now we can rewrite

$$\cos \omega(t - \tau) = \frac{\cos \omega(t - \tau_+) + \cos \omega(t - \tau_-)}{2 \cos \omega(\tau_+ - \tau_-)/2}$$

so that equation (3.7) can be interpreted as the superposition of two waves traveling, respectively, inward and outward with respect to the unperturbed flow, with wavenumbers, respectively, $k_{\pm} = \omega\tau'_{\pm} = \omega/(v \pm c)$ and velocities $v \pm c$, the superposition being modulated by the factor $(g/2) \cos [\omega \int^r c\, dr/(v^2 - c^2)]$. Even though we may thus describe each disturbance as the superposition of inward- and outward-traveling waves, one easily verifies that either kind on its own *cannot* satisfy the wave equation. This corresponds to the fact that the velocity of sound changes with r so that there is reflection at every r.

Taking account of equation (3.9a) we find eventually

$$\frac{1}{v} \frac{\partial}{\partial r} \left[v(v^2 - c^2) \frac{\partial f_1}{\partial r} \right] - \frac{c^2}{v^2 - c^2} \frac{\partial^2 f_1}{\partial s^2} = 0$$

which has in fact solutions of the form $g(r; \omega) \cos (\omega s)$, where g satisfies

$$[v(v^2 - c^2)g']' + \omega^2 \left(\frac{vc^2}{v^2 - c^2} \right) g = 0 . \tag{3.10}$$

This is a Sturm-Liouville equation for $g(r; \omega)$. Given a set of homogeneous boundary conditions $\alpha_i g + \beta_i g' = 0$ at $r = r_i$ $(i = 1, 2)$ (where the interval $[r_1, r_2]$ may not contain the sonic point), equation (3.10) has a denumerable set of eigenvalues ω_n. If $r_2 = \infty$, the spectrum ω_n is continuous, and summations over n have to be replaced by integrations. We will limit ourselves to the case where $r_2 < \infty$. The eigenfunctions g_n corresponding to ω_n are mutually orthogonal and quadratically integrable, and they form a basis in which any function satisfying the same boundary conditions can be expanded. The orthonormality relation is

$$\int_{r_1}^{r_2} \frac{vc^2}{v^2 - c^2} g_n g_m\, dr = \delta_{nm} . \tag{3.11}$$

In principle now one could write the complete solution of equation (3.1a) in the form

$$f_1(r, t) = \sum_{n=0}^{\infty} a_n g_n(r) \cos \omega_n[t - \tau(r) - \tau_{0n}] , \tag{3.12}$$

where the integration constants τ_{0n} were introduced explicitly because they may be different for each n. Unfortunately there are two problems which inhibit the use of this expansion in practice:

1. As far as we can presently see the eigenfunctions of equation (3.10) can be obtained only numerically;
2. While the change of variables (3.8) made the p.d.e. separable, it makes the initial and boundary conditions nonseparable, that is, a given set of boundary conditions imposed on the solution of equation (3.1a) cannot always be translated into a proper set of boundary conditions on $g(r)$ {e.g., the condition that $f'_1(r, t) = 0$ at r_i would lead to $g'(r) \cos \omega[t - \tau(r)] - \omega g(r)\tau'(r) \sin \omega[t - \tau(r)] = 0$ which is a time-dependent expression}.

In a number of cases, the latter difficulty may be overcome as follows. Suppose that a perturbation has been specified by its initial state in the form

$$f_1(r, t_0) = \varphi(r) , \qquad \dot{f}_1(r, t_0) = \psi(r) . \tag{3.13}$$

One can determine a set of basis functions $g_n(r; \omega_n)$ subject to the same conditions as $f_1(r, t)$; then at least one is sure that $f_1(r, t)$ does lie entirely in the Hilbert space spanned by the set $\{g_n(r; \omega_n) \mid n = 1, \dots \infty\}$, even if the individual terms in the expansion (3.12) do not. Subsequently one can project the expansion (3.12) onto the space spanned by the functions $g_n(r; \omega_n)$. Of course one cannot be sure that the projections of the functions $g_n(r) \cos \omega_n[t_0 - \tau(r) - \tau_{0n}]$ will be all linearly independent (which is the weakness we cannot avoid), but if they are, then the coefficients a_n in equation (3.12) can be obtained in the following way. Define

$$(a \mid b) = \int_{r_1}^{r_2} a(r)b(r) \frac{vc^2}{v^2 - c^2} \, dr , \tag{3.14}$$

and let

$$\varphi_l = (\varphi \mid g_l) , \qquad \psi_l = (\psi \mid g_l) , \tag{3.15a}$$

$$P_{mn} = [g_m \mid g_n \cos \omega_n(t_0 - \tau_{0n} - \tau)] , \tag{3.15b}$$

$$Q_{mn} = -\omega_n[g_m \mid g_n \sin \omega_n(t_0 - \tau_{0n} - \tau)] . \tag{3.15c}$$

Next arrange the numbers φ_l, ψ_l, a_l, P_{mn}, Q_{mn} in the columns Φ, Ψ, A, and the matrices P, Q, respectively. Then we find from equation (3.12)

$$\Phi = PA , \qquad \Psi = QA . \tag{3.16}$$

The integration constants τ_{0n} are determined by the condition that

$$P^{-1}\Phi = Q^{-1}\Psi \tag{3.17a}$$

and finally the expansion coefficients a_n are given by

$$A = P^{-1}\Phi . \tag{3.17b}$$

If this procedure happens to be inadmissible in a given case (i.e., if the projections of the individual terms of eq. [3.12] are not all linearly independent), this will result in the operator P having no inverse. If the boundary conditions are not properly satisfied by the solution (3.12) obtained in this way, then the latter will remain valid in a given point only for as long as no sound waves from the boundary have been able to reach this point.

3.3. *Transonic Flow*

Garlick (1979) established the stability of transonic (Bondi) accretion flow; Parker (1966) demonstrated the stability of transonic winds in the isothermal case (but see the comments of PSO on his treatment); PSO showed that both subsonic and transonic polytropic accretion are stable with respect to high-frequency perturbations (and it is easily seen that their results hold for outflow as well), but they did not comment on the effect of low-frequency perturbations and pointed out that the stability of "shocked" steady states was still conjectural. Although these results are not contradictory, there still seems to be some confusion and incompleteness in the arguments used; for instance Garlick's argument included rejecting solutions of equation (3.1) which become singular at the sonic point, while PSO saw no problem at all there. We shall now clear up this issue and demonstrate that (barring the shocked states meant by PSO, which will be discussed in the next section) the stability of transonic polytropic flow depends only on the conditions prevailing at the upstream end of the subsonic region.

The sonic point is undoubtedly a singular point for the perturbation equation (3.1), so one has to consider the subsonic region separately from the supersonic one.

In order to apply the criterion (3.6) to the subsonic region one has to take the sonic point r_s as one of the boundaries r_i in equation (3.5), so we first have to discuss the behavior of the solutions of equation (3.1b) in the vicinity of the sonic point. For a given value of ω the equation has two linearly independent ("fundamental") solutions; all other solutions are linear combinations of these. In order to find an expansion for these, we substitute

$$h(r) = (r - r_s)^\alpha[h_0 + h_1(r - r_s) + \cdots] \tag{3.18}$$

into equation (3.1b), expand the coefficients in this equation likewise, using equation (2.27), and collect the terms with equal powers of $(r - r_s)$; as a result we find the two allowed values for α:

$$\alpha = \begin{cases} 0 \\ -i\omega r_s \end{cases} \tag{3.19}$$

and a set of recursion relations for h_n, $n = 1, 2, \dots$ One of the fundamental solutions is regular ($\alpha = 0$) and travels smoothly through the sonic point, while the other one ($\alpha = -i\omega r_s$) is singular at $r = r_s$, more precisely: its value is finite but indefinite, while its derivative diverges here. Since

$$(r - r_s)^{-i\omega r_s} = e^{-i\omega r_s \log|r - r_s| + \pi\omega r_s H(r_s - r)} , \tag{3.20}$$

where $H(x)$ is the Heaviside function, it is easily seen that $(r - r_s)^{-i\omega r_s}e^{-i\omega t}$ represents a wave which travels away from the sonic point at both sides of it; at the subsonic side it moves upstream, with a velocity and wavelength which tend to zero at r_s; at the supersonic side it also moves upstream with respect to the fluid, but since $v > c$ here, it moves downstream with respect to an observer at rest. Anyway the "singular" solution does not cross the sonic barrier, so the singularity at r_s of equation (3.18) with $\alpha = -i\omega r_s$ is no reason to reject it as a possible solution. It is to be noted, moreover, that the "motion" of this solution is unidirectional *only* in the limit $r \to r_s$.

Returning now to the question of stability, we clearly see from equations (3.18) and (3.19) that $|h|^2$ is a regular function of r even if $\alpha = -i\omega r_s$, so that the contribution from the sonic point in equation (3.5) always vanishes. Therefore the stability in a subsonic region bounded at one end by the sonic point in fact depends only on the conditions at the other boundary of this region. So there is no need (nor any obvious justification, as far as we can see) to dispose of the solution which has a singularity at r_s, as suggested by Garlick (1979).

For the supersonic region obviously the same argument can be given, but here we can do even better. If the supersonic region is finite (Bondi accretion, or outflow ending in a shock), then Garlick's argument that any disturbance is carried away in a finite time, ensures the stability of the flow in this region. In the case of supersonic outflow extending to infinity, it follows from equation (2.20) that $\lim_{r\to\infty} c = 0$, $\lim_{r\to\infty} v = [2/(\gamma - 1)]^{1/2} \equiv v_\infty$. So the coefficients in equation (3.1b) become constants, and one easily sees that as $r \to \infty$ the solutions of equation (3.1b) must take the asymptotic form $h \simeq e^{-i\omega r/v_\infty}$ so that $\lim_{r\to\infty}(|h|^2)' = 0$. Since the contribution from the sonic point in equation (3.5) vanishes as well, we have $Q = 0$ so that the stability criterion is satisfied.

3.4. *Shocked Flow*

Consider now a flow pattern consisting of a supersonic region $[r_s, r_{sh}]$ and a subsonic region $[r_{sh}, r_{end}]$, separated by a steady adiabatic shock; we assume that the gas at either side of the shock is governed by a polytropic equation of state with the same value for the polytropic exponent γ. Here the criterion (3.6) cannot be applied since such a shock is bound to move if the flow variables are perturbed, so that we have no fixed intervals to consider (as assumed implicitly in the derivation of eq. [3.6]). Therefore we have to give a separate argument for this case:

1. A disturbance in the supersonic region is transmitted completely through the shock (see, e.g., Landau & Lifshitz 1963) so Garlick's argument remains valid in the case of shocked flow. One should notice, though, that the relative amplitude of the perturbation (e.g., $\delta p/p$) is amplified in the transmission, by a factor depending on the Mach number; in extreme cases therefore a small perturbation on the supersonic side could result in a strong one on the subsonic side, for which the linear perturbation calculation is not valid.
2. A disturbance in the subsonic region is reflected at the shock and the reflected amplitude is weaker than the incident one. Thus for the subsonic region the shock front acts like a partially absorbing boundary so that it cannot be responsible for any instability in the subsonic flow.

However, there is another point to consider: since the shock position will in general oscillate due to a perturbation (see, e.g., Landau & Lifshitz 1963), the question arises whether the amplitude of such oscillations may be large and, if not, whether there may nevertheless be an appreciable cumulative displacement after many small oscillations. The actual motion of the shock will be discussed in a forthcoming paper from a more general point of view, but we cite here the following preliminary results which were derived for a flow with uniform density:

1. The amplitude of the oscillatory motion of a plane shock front perturbed by a linear sound wave may be comparable to the wavelength of the disturbance or to its spatial extent if it is a localized wavepacket, but the net displacement which remains after a disturbance has been transmitted (coming from the supersonic side) or reflected (coming from the subsonic side) decreases with increasing Mach number of the shock and is mostly several orders of magnitude smaller, except when a relatively weak shock (Mach number less than, say, two or three) is hit by a perturbation from the subsonic side; in the latter case the results from our linear approximation suggest the possibility of a much stronger effect which, however, will require a nonlinear treatment;
2. The net displacement is always in the direction of motion of the incident disturbance (i.e., upstream for a perturbation on the subsonic side and downstream on the other side).

In the case of spherical flow the above results should remain valid at least qualitatively, since an adiabatic shock may occur anywhere in a supersonic flow (see § 2.4) so that there is no direct impediment to its motion; on the other hand we must bear in mind that the flow is nonuniform in our case and that a shock moving over a distance comparable to the scale length of this nonuniformity might force the flow into a nonstationary state. A more thorough discussion of the motion of a shock through spherical polytropic flow will be given elsewhere.

Returning now to the question of stability we come to the following conclusion: barring very special boundary conditions at the end of the subsonic region, the shocked steady state is stable in the sense that the amplitude of perturbations does not grow in time; the position of a strong shock may oscillate under the influence of small perturbations, but it should remain stationary on average; however, if the shock is relatively weak, the flow must be regarded as unstable in the sense that small perturbations incident from the subsonic side may cause the shock front to migrate upstream over a considerable distance, particularly if there are many consecutive disturbances (however small) because the net displacement due to one brings the shock somewhat closer to the sonic point where its Mach number is smaller so that the next disturbance will be the more effective.

4. SUMMARY

We have derived the complete set of solutions to the problem of stationary, spherically symmetric polytropic flow around a star. These solutions are the different branches of a quartic equation which turns out to have at most two real roots. If the radiation pressure force from the star does not dominate its gravitational pull, the nature of the branches is different according as the dimensionless flow rate is larger or smaller than a critical value λ_{max} which depends only on the polytropic exponent γ. For $\lambda < \lambda_{max}$ there are two real branches, one of which describes supersonic flow (Mach number $M \to \infty$ both as $r \to 0$ and $r \to \infty$), while the other describes subsonic flow (with $M \to 0$ both as $r \to 0$ and $r \to \infty$). The latter may describe subsonic accretion. For the critical value $\lambda = \lambda_{max}$ these two branches can be matched continuously at the sonic point (where $M = 1$ for both); a flow regime which is subsonic at its base may switch to the other branch there and become supersonic, thus describing either transonic outflow of the solar wind type or Bondi (Type II) accretion. For $\lambda > \lambda_{max}$ these branches are complex between two radii from the star; they cannot describe continuous stationary flow in this case, but they may be reached from a supersonic flow regime via an adiabatic shock which can occur in the supersonic flow at any distance from the star.

If the radiation pressure dominates the gravitational force (super-Eddington flow) there is a physically relevant branch for any value of λ but only beyond a critical distance $r = r_E$ from the star; it consists of two parts which describe, respectively, subsonic and supersonic outflow and which are matched at $r = r_E$ where both have $M = 1$; the subsonic part can only be relevant in a shocked flow regime.

We derived a stability criterion for these flows, for linear, spherically symmetric perturbations. This criterion is valid for all wavelengths (this is possible since, as the flow is not selfgravitating, the Jeans length is infinite). The amplitude of the perturbation satisfies a Sturm-Liouville equation. For both subsonic and transonic flow, stability is determined by the conditions prevailing at the star. The occurrence of a shock in general does not change our conclusions. However, the result of perturbations on a shocked flow is a migration of the shock front if the shock is not very strong ($M < 3$, say). This may ultimately change the flow pattern quite drastically.

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APPENDIX

For the isothermal flow case, $\gamma = 1$, we choose f/c^2 and c as our unit of distance and velocity. We write the constant in equation (2.5b) as $c^2 \ln \rho_0$, and use ρ_0 as our unit of density. We then find:

$$r^4 - \frac{1}{\ln \rho} r^3 = -\frac{1}{2} \frac{\lambda^2}{\rho^2 \ln \rho} \tag{A1}$$

instead of equation (2.9) with

$$\lambda \equiv \frac{Ac^3}{4\pi f^2 \rho_0} \tag{A2}$$

instead of equation (2.8). We will use ρ as our independent variable. Proceeding as in the case $\gamma > 1$, we find the solution (eqs. [2.15]–[2.16]), with the replacements

$$R = -\frac{16\lambda^2}{a^3 \rho^2}, \tag{A3a}$$

$$a = \frac{1}{4 \ln \rho}. \tag{A3b}$$

Equation (2.14) is replaced by $\lambda_{max}^2(\gamma = 1) = e^3/16$. For the asymptotic behavior of equations (2.15)–(2.16) and (A3), we find for $\rho \to 0$, 1, and ∞:

1. For the subsonic branch,

$$r_1^+ \to \frac{1}{\ln \rho} \to 0 \text{ for } \rho \to \infty, \tag{A4a}$$

$$r_1^+ \to \frac{1}{\rho - 1} \to \infty \text{ for } \rho \to 1. \tag{A4b}$$

2. For the supersonic branch,

$$r_1^- \to \left(\frac{r_A}{\rho}\right)^{2/3} \to 0 \text{ for } \rho \to \infty\ , \tag{A5a}$$

$$r_1^- \to \frac{(8\lambda^2)^{1/4}}{(-\ln\rho)^{1/4}\rho^{1/2}} \to \infty \text{ for } \rho \to 0\ ; \tag{A5b}$$

3. For the supersonic, super-Eddington branch,

$$r_1^+ \to -\frac{(8\lambda^2)^{1/4}}{(-\ln\rho)^{1/4}\rho^{1/2}} \to -\infty \text{ for } \rho \to 0\ , \tag{A6a}$$

$$r_1^+ \to r_E\left[1 + \frac{2\rho_E}{\lambda^2(3 - 2\ln\rho_E)}\left(\frac{\rho - \rho_E}{\rho_E}\right)^2\right] \to r_E\ . \tag{A6b}$$

Here, r_A is given by equation (2.23f), and (r_E, ρ_E) is the simultaneous solution of equation (A1) and $r = -(\lambda/\rho)^{1/2}$. Equation (A6b) is the behavior of the super-Eddington branch close to the base of the flow.

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